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Area Preserving Diffeomorphisms and 2-d Gravity

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Area preserving diffeomorphisms of a 2-d compact Riemannian manifold with or without boundary are studied. We find two classes of decompositions of a Riemannian metric, namely, h- and g-decomposition, that help to formulate a gravitational theory which is area preserving diffeomorphism (SDiffM-) invariant but not necessarily diffeomorphism invariant. The general covariance of equations of motion of such a theory can be achieved by incorporating proper Weyl rescaling. The h-decomposition makes the conformal factor of a metric SDiffM-invariant and the rest of the metric invariant under conformal diffeomorphisms, whilst the g-decomposition makes the conformal factor a SDiffM scalar and the rest a SDiffM tensor. Using these, we reformulate Liouville gravity in SDiffM invariant way. In this context we also further clarify the dual formulation of Liouville gravity introduced by the author before, in which the affine spin connection is dual to the Liouville field.

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1. Introduction

The geometry of compact oriented manifold is one of the key ingredients to study gravitational theories. It is also a very useful tool to investigate certain two dimensional physics. In traditional approaches of gravity we require a theory is covariant under diffeomorphisms which are customarily called the general coordinate transformations with respect to a local coordinate system. There is additional symmetry in the frame (vielbein) space which is called the local Lorentz symmetry. In two-dimensions extra information is needed because we often are led to work with conformal geometry that allows changes of metric distances, which is a less restrictive geometry compared to the usual Riemannian geometry. This extra information is provided by Weyl rescalings which change conformal factor of a metric and are not necessarily achieved by (conformal) diffeomorphisms. For the Euclidean signature, this is effectively described by the conformal geometry of Riemann surfaces. Of course, it is not really essential to require Weyl invariance of a theory, but 2-d theories often happen to be Weyl invariant.

The general covariance of a gravitational theory is rooted in the properties of diffeomorphism group $\text{Diff}M$ of manifold M on which the theory is defined and fundamental variables are covariant objects with respect to $\text{Diff}M$. In this paper we shall attempt to provide a framework to formulate a theory in which dynamical variables are not covariant objects with respect to $\text{Diff}M$ but behave covariantly under smaller symmetry: volume preserving diffeomorphisms. These are diffeomorphisms that leave a given volume element invariant and they form a subgroup $\text{SDiff}M$ of $\text{Diff}M$. $\text{SDiff}M$ includes isometry group so that particularly in the flat case the generators of the Poincaré group satisfies the volume preserving condition. In 2-d these are usually called area preserving diffeomorphisms for an obvious reason.

The relevance of the area preserving diffeomorphism group $\text{SDiff}M$ can be easily appreciated in the string theory based on the Nambu-Goto action[1], whose lagrangian density is just the area element of a given surface without specifying any intrinsic metric, so that $\text{SDiff}M$ is the fundamental symmetry for both open and closed strings. Such lack of manifest covariance (also the non-linearity of the lagrangian) is usually regarded as a set-back of Nambu-Goto's approach to string theory. In the Polyakov string theory this is enlarged to the world-sheet $\text{Diff}M$ and the Weyl rescaling[2], nevertheless the area preserving structure resurfaces in non-critical dimensions through Liouville modes. Furthermore, $\text{SDiff}M$ completely excludes any conformal diffeomorphisms so that one can identify the genuine dilaton, which in principle is supposed to incorporate all the degrees of freedom associated with rescaling of metric. Perhaps,

this may indicate that full understanding of the role of $\text{SDiff}M$ might be the key to understand the Liouville modes of noncritical string theories as well as the dilaton. Therefore, it is quite tempting to contemplate on the role of $\text{SDiff}M$ more seriously.

In [3] the author formulates a manifestly area preserving diffeomorphism invariant gravitational theory, taking analogy of the hydrodynamics of incompressible, ideal fluids[4]. The main idea is to take $\text{Diff}M$ as configuration space of two-dimensional surfaces (as Riemannian manifolds, not as Riemann surfaces), then to require $\text{SDiff}M$ as underlying symmetry. This is different from conventional approaches in which we take the space of all metrics with $\text{Diff}M$ as underlying symmetry. This is certainly reasonable at least on genus zero surfaces because all metrics are related by diffeomorphisms. For higher genus surfaces, the space of all metrics is bigger than $\text{Diff}M$ due to the Teichmüller deformations, so to apply such a scheme it is inevitable to enlarge the configuration space. It turns out that classically the theory has an equivalent form of action to Liouville gravity as an induced gravity[5, 6], although the underlying structures are different. Then it is suggested that the quantum theory might be different because of potentially different quantization due to the area preserving structure.

We can in fact decompose $\text{Diff}M \simeq \mathcal{V} \otimes \text{SDiff}M \supset \mathcal{W}_c \otimes \text{SDiff}M$ and show that the Liouville gravity action can be rewritten in terms of $\sigma_{\mu\nu} = e^{-2\phi} g_{\mu\nu}$, where ϕ and $\sigma_{\mu\nu}$ are not necessarily covariant objects with respect to $\text{Diff}M$, such that $\mathcal{S}_L(\phi, \sigma)$ is $\text{SDiff}M$ -invariant but not manifestly $\text{Diff}M$ -invariant. Here, \mathcal{V} is the space of all volume elements with an equal volume and \mathcal{W}_c is the space of all conformal diffeomorphisms. It is also possible that we take $\text{SDiff}M$ as underlying symmetry but enlarge the configuration space to the space of all metrics. Then the approach taken in [3] also turns out to provide a dual way of formulating Liouville gravity with $\text{SDiff}M$ as a gauge symmetry. This will be further pursued in this paper.

Lately, a certain “area preserving” structure in the Liouville gravity was also investigated in [7, 8]. The main idea is based on the fact that, from the conformal geometry’s point of view, the amount of gauge degrees of freedom provided by $\text{Diff}M$ is equivalent to the amount provided by the combination of the Weyl rescaling and $\text{SDiff}M$. In [7, 8] however a coordinate-choice-dependent (*i.e.* non-covariant) condition of area preserving structure is used so that it inevitably restricts the jacobian of any coordinate change to be unity. In general, the diffeomorphisms whose jacobians are harmonic transform the Weyl-invariant part of $\sqrt{|g|}R$ still like a scalar density, so it is still true if the jacobian is a harmonic function. However, as a result, the action is not a well-defined integration on a manifold because it depends on a local coordinate basis. In general, a noncovariant condition cannot be imposed globally on a curved manifold.

In this paper we shall find that in fact a covariant condition can be imposed and a similar argument can be still followed. The key observation is that the conformal diffeomorphisms \mathcal{W}_c can be trade off with Weyl rescalings. The integration can be well defined in terms of a gauge transformation whenever it is necessary and equations of motion. This is because a noncovariant object is usually not globally defined and it depends on a gauge.

In general, we can think of two different cases of $\text{SDiff}M$ -invariant theories: First, a lagrangian is written in terms of usual covariant objects and it is defined on a manifold with boundary. Second, a lagrangian is not written in terms of covariant objects and is defined on a manifold with or without boundary. The key idea is that fundamental fields are no longer covariant objects under $\text{Diff}M$, but they are covariant with respect to $\text{SDiff}M$, although not necessarily globally defined over M . In both cases action must be a well-defined integration on M at most up to a gauge transformation with respect to changes of local coordinate basis.

The first case is a very modest modification. If $\partial M = 0$, $\int_M \hat{R}$ is invariant under $\text{Diff}M$, being a topological invariant. However, if $\partial M \neq 0$, $\int_M \hat{R}$ is no longer invariant under $\text{Diff}M$ but picks up a boundary term, although it still is independent from the choice of local coordinate basis. When we have to deal with a gravitational theory in which such boundary is relevant, we are required to introduce an extra surface term that normally depends on the extrinsic (geodesic) curvature to preserve the diffeomorphism invariance of the theory. A variation of metric induces a variation of a surface term which is required to vanish to derive equations of motion. So, strictly speaking, boundary terms modify equations of motion too. Thus an explicit boundary term is added to derive the same equations of motion as in the case without boundary, if a gravitational theory is defined on a manifold with boundary. In fact quantum gravity in the path integral formalism is usually such a case[9]. Suppose we required that a theory were only invariant under $\text{SDiff}M$, we would not be obliged to introduce such a surface term because the boundary term actually vanishes. Thus we could speculate that, if physics near a boundary might break the general covariance, we reduce the symmetry to a smaller $\text{SDiff}M$ covariance, instead of introducing a surface term to restore the general covariance everywhere.

In fact this is quite generic. Any theory defined on a curved manifold can be interpreted this way. Either we introduce a surface term to preserve the general covariance, or we could restrict to the volume preserving diffeomorphism invariance. Of course, whether this is a legitimate thing to do is another question. We have no intention to abandon the principle of general covariance which we learned from Einstein's theory, but we can still try to see how a theory can be formulated without manifest diffeomorphism invariance everywhere but only

with manifest volume preserving diffeomorphism invariance. It turns out that in certain cases the variational principle based on $\text{SDiff}M$ merely changes the cosmological constant due to the Bianchi identity. Thus it still maintains the general covariance at the level of equations of motion. Also in the flat limit it still does not contradict to the usual Poincaré symmetry because Poincaré transformations are isometric and isometric diffeomorphisms are volume-preserving.

The second case is more drastic. Usually, a lagrangian of this type is written in terms of a metric-like object that does not transform like a tensor. Nevertheless, under $\text{SDiff}M$ it has a well-defined transformation property and that the action defined by integrating such a lagrangian over M is $\text{SDiff}M$ -invariant. Weyl rescaling in terms of a proper object, we can rewrite equations of motion in terms of a metric tensor and the general covariance can be still recovered away from the boundary.

This paper is organized as follows: In section two some general properties of diffeomorphism group and volume preserving diffeomorphism group are explained. The effects of diffeomorphisms are described in coordinate-independent (*i.e.* active) way. In section three we study the role of area preserving diffeomorphisms in the Liouville gravity. We also clarify the approach in [7, 8]. In section four a dual formulation of Liouville gravity in terms of frame introduced in [3] is further investigated. The gauge fixing condition of local Lorentz symmetry in the frame space motivated by the area preserving diffeomorphism is analyzed in detail. This gauge fixing condition is the key to relate to the Liouville gravity. Also, some remarks on the zeroth order formalism are given. Finally, in section five we give comments on the generic structure of gravitational theories with area preserving diffeomorphism group and discuss the relevance of the issues presented in this paper.

2. $\text{Diff}M$ and $\text{SDiff}M$

2.1. $\text{Diff}M$

For a compact oriented manifold M , the diffeomorphism group ¹, $\text{Diff}M$, is an infinite-dimensional Lie group which consists of \mathbf{C}^∞ -diffeomorphisms $f : M \rightarrow M$ [10].

In this paper we shall adopt an *active* way of describing diffeomorphisms:

$$f^* : g \rightarrow g^{(f)}. \quad (2.1)$$

¹We shall be interested in the orientation preserving case only so that we shall denote $\text{Diff}M$ to be Diff^+M .

In terms of local coordinates, particularly the elements connected to the identity, i.e. $f \in \text{Diff}_0 M$, can be written infinitesimally as

$$f^\mu(x^\alpha) = x^\mu + \epsilon v^\mu(x^\alpha), \quad (2.2)$$

where $v = v^\mu \partial_\mu \in \text{Vect} M$ is a vector field on M and ϵ is an infinitesimal parameter. v 's form a Lie algebra so that we denote $v \in \text{diff} M$. We can express the change of a metric under $\text{Diff}_0 M$ infinitesimally (for example, see [11]) as

$$f^* : ds^2 = g_{\mu\nu} dx^\mu dx^\nu \mapsto ds_f^2 = g_{\mu\nu}^{(f)} dx^\mu dx^\nu = (g_{\mu\nu} + \delta_f g_{\mu\nu}) dx^\mu dx^\nu = g_{\mu\nu}(f) df^\mu df^\nu. \quad (2.3)$$

Then $\delta_f g_{\mu\nu}$ is nothing but the Lie derivative defined by one-parameter subgroup of $\text{Diff}_0 M$:

$$\mathcal{L}_v g_{\mu\nu} = \delta_f g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu, \quad (2.4)$$

where ∇_μ is the covariant derivative with respect to the Riemannian connection. The tensorial property of the metric which is also useful in the *passive* approach is given by incorporating metric form changes as well as coordinate changes on an overlap of coordinate charts such that

$$\tilde{g}_{\mu\nu}(f) \frac{\partial f^\mu}{\partial x^\alpha} \frac{\partial f^\nu}{\partial x^\beta} = g_{\alpha\beta}. \quad (2.5)$$

Weyl rescalings are changes of a metric $g \rightarrow e^{2\phi} g$ that are not necessarily accomplished by diffeomorphisms.

More precisely, now let us clarify the relation between the active approach and the passive approach by explicitly comparing them. Note that in the above $\tilde{g}_{\mu\nu}(f)$ is not the same as $g_{\mu\nu}^{(f)}(x)$. Infinitesimally, eq.(2.5) can be expanded according to eq.(2.2) as

$$\tilde{g}_{\alpha\beta}(x) + \epsilon (v^\sigma \partial_\sigma \tilde{g}_{\alpha\beta} + \tilde{g}_{\mu\beta} \partial_\alpha v^\mu + \tilde{g}_{\mu\alpha} \partial_\beta v^\mu) = g_{\alpha\beta}(x). \quad (2.6)$$

Since the form change between $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ should be of order ϵ infinitesimally, in the second term of the LHS we can replace $\tilde{g}_{\alpha\beta}$ with $g_{\alpha\beta}$. Thus we recover

$$g_{\mu\nu} - \tilde{g}_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu = \delta_f g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu} \quad (2.7)$$

At least in the leading order of ϵ the difference between eq.(2.3) and eq.(2.5) is whether one defines in an *active* way or in a *passive* way, which is reflected by the sign of the change in the metric form.

At this moment we would like to call the reader's attention to the fact that the coordinate invariance of an object is not necessarily the same as the diffeomorphism invariance of the object. If an object satisfies a vanishing Lie derivative, it is said to be $\text{Diff}M$ -invariant. For tensors, this implies in particular the tensorial form invariance. For example, a metric tensor is not $\text{Diff}M$ -invariant, but invariant only under isometries. In particular, at the level of equations of motion $\text{Diff}M$ -invariance is equivalent to the general covariance. But at the level of an action $\text{Diff}M$ invariance is a stronger statement than the invariance under coordinate transformations because a coordinate transformation is merely a change of coordinate basis. It is absolutely necessary for an action to be independent from a choice of local coordinate basis to be a well-defined integration on M .

For a scalar $S(x)$ we know that under the general coordinate transformation it should transform like $\tilde{S}(f) = S(x)$, but $\mathcal{L}_v S$ (which can be expressed locally as $v^\mu \partial_\mu S$) does not necessarily vanish. The former incorporates coordinate change as well as a form change. However an infinitesimal diffeomorphism ignores the coordinate change but measures the form change only. These two are not the same in general for other tensors either. A lagrangian density has a form of $\sqrt{|g|}S$, where $g = \det g_{\mu\nu}$. Under coordinate transformation, this transforms like a scalar density. Under $\text{Diff}_0 M$, $\mathcal{L}_v(\sqrt{|g|}S) = \partial_\mu(v^\mu \sqrt{|g|}S)$. These two changes are not the same even infinitesimally because the jacobian is given by $J = 1 + \partial_\mu v^\mu$.

Note that the equality between volume elements in two-dimensions

$$d^2 f \sqrt{|g'(f)|} = d^2 x \sqrt{|g(x)|} \quad (2.8)$$

is always true for any f as a coordinate transformation, which can be shown easily using jacobian. This simply means a volume element does not depend on a choice of a local coordinate basis and it is also necessarily for a volume element to be well-defined on a curved manifold. There is another way to check this invariance without using the jacobian. The LHS of eq.(2.8) can be expanded in terms of $f = x + v$ as

$$d^2 f \sqrt{|g'(f)|} = d^2 x \sqrt{|g'(x)|} (1 + \nabla_\mu v^\mu).$$

We can now recover the identity eq.(2.8) for any $f \in \text{Diff}M$, using the equality

$$d^2x\sqrt{|g'(x)|} = d^2x\sqrt{|g(x)|}(1 - \nabla_\mu v^\mu).$$

This simple computation without using jacobian is significant in the sense that all the ingredients are covariantly defined over manifold M without preferred choice of a coordinate system. Also this is a source to the confusion that the coordinate invariance is equivalent to the diffeomorphism invariance, which is not always true. To prove whether an action of noncovariant objects is a well-defined integration or not, this method is very useful.

We can compare each step to the diffeomorphism case. Under coordinate transformation $x \rightarrow f$ we obtain

$$\sqrt{|g'(f)|} - \sqrt{|g(x)|} = -\sqrt{|g(x)|}\partial_\mu v^\mu.$$

On the other hand, under diffeomorphism $f = x + v$ we have

$$\mathcal{L}_v \sqrt{|g(x)|} = \sqrt{|g(x)|}\nabla_\mu v^\mu.$$

Ignoring the sign difference, which merely reflects the active and the passive way of using the transformation, the difference of the above is $v^\mu \partial_\mu \sqrt{|g(x)|}$, which precisely measures the functional change with respect to the coordinate change. Thus from $\text{Diff}M$'s point of view eq.(2.8) is not a proper way to compare and the jacobian is not a good object to use either. Upon integration the coordinates are in fact dummy so that we should really compare $d^2x\sqrt{|g'(x)|}$ and $d^2x\sqrt{|g(x)|}$. Note that the equality $d^2x\sqrt{|g'(x)|} = d^2x\sqrt{|g(x)|}$ can only be achieved if $\mathcal{L}_v \sqrt{|g|} = 0$, i.e. $\nabla_\mu v^\mu = 0$, which is the condition to define area preserving diffeomorphisms.

2.2. SDiffM in general

Volume preserving diffeomorphisms are not necessarily characterized by the property of preserving volume itself because diffeomorphisms also preserve volume as we pointed out in the previous section. We need to require a stronger condition to distinguish them.

The **volume preserving conditions** are defined by the follows: For $f \in \text{Diff}M$ and $v \in \text{diff}M$,

- **[VP1]** $g^{\mu\nu}\delta_f g_{\mu\nu} = 0$, i.e. $\nabla_\mu v^\mu = 0$.
- **[VP2]** v^μ is tangential to the boundary ∂M .

If a vector field v satisfies the above conditions, then v 's form a Lie subalgebra $\text{sdiff}M$ of $\text{diff}M$ such that $[u, v] = w$ and $\nabla_\mu u^\mu = 0 = \nabla_\mu v^\mu$ implies $\nabla_\mu w^\mu = 0$. The corresponding f is called a volume preserving diffeomorphism. These vector fields generate a subgroup of $\text{Diff}M$ called volume preserving diffeomorphism group, $\text{SDiff}_{\hat{\mu}}M$ for a volume element $\hat{\mu}$. **[VP1]** is the condition which leaves this volume form invariant, particularly, $\delta_f \sqrt{|g|} = 0$ and **[VP2]** prohibits any area change over boundary from occurring. In terms of the codifferential δ , **[VP1]** becomes $\delta \hat{v} = 0$ for the one-form $\hat{v} = g_{\mu\nu} v^\mu dx^\nu$ corresponding to vector field v . The infinitesimal actions of Poincaré group actually satisfy **[VP1]**, being isometric.

For any vector V^μ on M , if $f \in \text{SDiff}M$, then

$$\nabla_\mu^{(f)} V^\mu = \nabla_\mu V^\mu, \quad (2.9)$$

where $\nabla_\mu^{(f)}$ is a covariant derivative in terms of $g_{\mu\nu}^{(f)}$. This identity is due to $\delta_f \Gamma_{\alpha\mu}^\alpha = 0$ if $f \in \text{SDiff}M$.

In terms of zweibeins e_μ^a and their inverses E_a^μ such that

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad g^{\mu\nu} = \eta^{ab} E_a^\mu E_b^\nu,$$

the torsion-free affine spin connection² $\omega_\mu^a{}_b$ is given by

$$\omega_\mu^a{}_b = -E_b^\alpha \nabla_\mu e_\alpha^a. \quad (2.10)$$

Using $\delta_f g_{\mu\nu} = \eta_{ab} (\delta_f e_\mu^a e_\nu^b + e_\mu^a \delta_f e_\nu^b)$, **[VP1]** now reads

$$g^{\mu\nu} \delta_f g_{\mu\nu} = 2E_a^\mu \delta_f e_\mu^a = 0. \quad (2.11)$$

This spin connection is not necessarily globally defined over M as there is no global frame over M . As is well known, it quite resembles a gauge theory.

Just to clarify the notation, the Lie derivative acting on a p-form α is read as

$$\mathcal{L}_v \alpha = (di_v + i_v d)\alpha, \quad (2.12)$$

²This notation can be potentially confusing in other than two-dimensions. Even in two-dimensions sometimes we need to keep in mind the hidden frame indices because they determine the transformation property of the object in the frame space.

where i_v is the inner product with respect to a vector field v . Since i_v lowers the rank of a differential form, in particular $i_v S = 0$ for a scalar S .

In fact in n -dimensions any n -form Ω satisfies that

$$\int_M \mathcal{L}_v \Omega = \int_M d i_v \Omega. \quad (2.13)$$

Unless v and Ω are globally defined on M , there is no obvious reason why $i_v \Omega$ is globally defined. However, let us assume it is globally defined to apply Stoke's theorem so that

$$\int_M \mathcal{L}_v \Omega = \int_{\partial M} i_v \Omega. \quad (2.14)$$

For $v \in \text{diff} M$ this does not necessarily vanish. However, if $v \in \text{sdiff} M$, the RHS vanishes because $v \parallel \partial M$. This property of $\text{SDiff} M$ is very important for us to incorporate volume preserving diffeomorphism as a symmetry of a given physical system defined on a manifold with boundary.

Finally, we quote one important theorem: Omori-Ebin-Marsden's theorem[12, 13]. It states that $\text{Diff} M$ is diffeomorphic to $\mathcal{V} \otimes \text{SDiff}_{\hat{\mu}} M$, where $\mathcal{V} = \{\hat{\nu}\}$ is the space of all volume elements that satisfy $\int_M \hat{\nu} = \int_M \hat{\mu}$. This theorem is not only true for manifold without boundary but also true for manifold with boundary.

2.3. Metric decompositions

Notice that conformal Killing vectors do not satisfy the volume preserving conditions. Using this property, we can separate $\text{SDiff} M$ completely from any conformal transformations. This can be done by decomposing the metric in following way. Let $g_{\mu\nu} = e^{2\phi} h_{\mu\nu}$, then we shall call it **h-decomposition** for future referencing purpose, if

$$\delta_f g_{\mu\nu} = e^{2\phi} \delta_f h_{\mu\nu} + h_{\mu\nu} \delta_f e^{2\phi},$$

where in n -dimensions

$$\delta_f h_{\mu\nu} = v^\alpha \partial_\alpha h_{\mu\nu} + h_{\mu\alpha} \partial_\nu v^\alpha + h_{\alpha\nu} \partial_\mu v^\alpha - \frac{2}{n} \nabla_\alpha^{(h)} v^\alpha h_{\mu\nu}, \quad (2.15)$$

$$\delta_f e^{2\phi} = \frac{2}{n} \nabla_\alpha v^\alpha e^{2\phi}. \quad (2.16)$$

For notational convenience we defined $\nabla_\alpha^{(h)}$ as a covariant derivative in terms of $h_{\mu\nu}$, although $h_{\mu\nu}$ does not necessarily transform like a metric tensor. Note that $h^{\mu\nu} \delta_f h_{\mu\nu} = 0$ for any

$f \in \text{Diff}M$. In this sense we can regard $h_{\mu\nu}$ as sort of a metric to $\text{SDiff}M$. This decomposition now clearly shows that $h_{\mu\nu}$ is invariant under any conformal diffeomorphisms, whilst $e^{2\phi}$ is invariant under $\text{SDiff}M$. Both variations can vanish at the same time only if f is an isometry. So we have explicitly separated the conformal factor from the rest and ϕ can be identified as a true dilaton because fixing ϕ fixes conformal degrees of freedom completely³. Now a Weyl transformation of $g_{\mu\nu}$ can be regarded as a change of ϕ so that the property of eq.(2.16) can be preserved.

If we have a theory of an action $\mathcal{S}(h_{\mu\nu}, \phi, \dots)$ in which $h_{\mu\nu}$ takes the role of a metric tensor and if it is invariant under eqs.(2.15) for $\nabla_\mu v^\mu = 0$, then the theory is $\text{SDiff}M$ -invariant. If \mathcal{S} does not contain ϕ , then it is also Weyl-invariant. A simple example is to take $\int_M \sqrt{|h|} R(h)$. It does not depend on ϕ at all so that we can say it is not manifestly $\text{Diff}M$ -invariant. Nevertheless, it can be shown that the action does not depend on a choice of local coordinate basis, so it is a well-defined integration on M . Under $\text{SDiff}M$ the action is invariant, if $v^\mu \partial_\mu \phi = 0$ on ∂M , which can be imposed as a boundary condition. One can also easily show that equations of motion become covariant ones with respect to $\text{Diff}M$ by a simple Weyl rescaling. In fact, this action is actually $\text{Diff}M$ -invariant if $\partial M = 0$ because the change is just a total derivative.

It turns out that the above h-decomposition is not the only one interesting. There is another important decomposition which we shall call **g-decomposition**. In the g-decomposition we can in fact define a metric tensor with respect to $\text{SDiff}M$ as we define $g_{\mu\nu}$ with respect to $\text{Diff}M$. In this case we let $g_{\mu\nu} = e^{2\hat{\phi}} \hat{g}_{\mu\nu}$ and that

$$\delta_f g_{\mu\nu} = e^{2\hat{\phi}} \delta_f \hat{g}_{\mu\nu} + \hat{g}_{\mu\nu} \delta_f e^{2\hat{\phi}},$$

where for some nonvanishing constant s

$$\delta_f \hat{g}_{\mu\nu} = v^\alpha \partial_\alpha \hat{g}_{\mu\nu} + \hat{g}_{\mu\alpha} \partial_\nu v^\alpha + \hat{g}_{\alpha\nu} \partial_\mu v^\alpha - s \nabla_\alpha v^\alpha \hat{g}_{\mu\nu}, \quad (2.17)$$

$$\delta_f e^{2\hat{\phi}} = v^\alpha \partial_\alpha e^{2\hat{\phi}} + s \nabla_\alpha v^\alpha e^{2\hat{\phi}}. \quad (2.18)$$

Although under $\text{Diff}M$ they behave in quite unusual way, but under $\text{SDiff}M$

$$\hat{\mathcal{L}}_v \hat{g}_{\mu\nu} := \delta_f \hat{g}_{\mu\nu} = v^\alpha \partial_\alpha \hat{g}_{\mu\nu} + \hat{g}_{\mu\alpha} \partial_\nu v^\alpha + \hat{g}_{\alpha\nu} \partial_\mu v^\alpha, \quad (2.19)$$

$$\hat{\mathcal{L}}_v e^{2\hat{\phi}} = \delta_f e^{2\hat{\phi}} = v^\alpha \partial_\alpha e^{2\hat{\phi}} \quad (2.20)$$

³Fixing these degrees of freedom should generate an independent scale in a theory. It is shown in [14] that the dilaton indeed generates a scale which is independent from the Newton's constant in terms of its own scale parameter in two-dimensions.

so that we can use $\hat{g}_{\mu\nu}$ to build manifestly $\text{SDiff}M$ -invariant actions. Due to the area preserving condition, eq.(2.19) actually depends on $\hat{\phi}$. In this case, since a Weyl transformation does not preserve the structure of eq.(2.18), the action of the form $\mathcal{S}(\hat{g})$ is not necessarily Weyl invariant.

Just to make the story complete, we include $g_{\mu\nu} = e^{2\varphi}\tilde{g}_{\mu\nu}$ such that φ transforms like a scalar and $\tilde{g}_{\mu\nu}$ transforms like a tensor under $\text{Diff}M$. Then $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are related by a conformal diffeomorphism or a Weyl rescaling. Comparing to the h-decomposition, we can see why fixing φ does not really fix all the conformal degrees of freedom. This is because there still are conformal degrees of freedom in the trace of the metric variation. We get complete fixing in the h-decomposition and then the remaining symmetry is $\text{SDiff}M$. Finally, just to summarize, we have the following identity: $g_{\mu\nu} = e^{2\phi}h_{\mu\nu} = e^{2\hat{\phi}}\hat{g}_{\mu\nu} = e^{2\varphi}\tilde{g}_{\mu\nu}$.

2.4. Symplectic structure of $\text{SDiff}M$ in 2-d

Now we can easily check that the volume element $\hat{\mu} = e^1 \wedge e^2$ is indeed invariant under area preserving diffeomorphisms in coordinate-independent manner as follows:

$$\mathcal{L}_v \hat{\mu} = \text{di}_v(e^1 \wedge e^2) = - * \delta \hat{v} = -\hat{\mu} \delta \hat{v} = 0. \quad (2.21)$$

This also implies that in 2-d we have a symplectic manifold $(M, \hat{\mu})$ and $v \in \text{sdiff}M$ is nothing but a hamiltonian vector field on M . $\text{SDiff}M$ is a Lie group acting on this symplectic manifold. Thus locally we have

$$i_v \hat{\mu} = -\text{d}H$$

for some function H . $\mathcal{L}_v H = 0$ implies that $\text{SDiff}M$ is a group of symmetries of this Hamiltonian system. The appearance of such a symplectic structure is a unique property in two-dimensions so that there might be an interesting hamiltonian formalism of 2-d gravity motivated by this.

2.5. Generators

Note that $\text{Diff}S^1$ is related to \mathcal{W}_c and is not a subgroup of $\text{SDiff}M$. Hence, as far as 2-d gravity is concerned, one can have a hope that $\text{SDiff}M$ may contain information that $\text{Diff}S^1$ lacks, but is helpful to better understand 2-d gravity. Therefore, explicit forms of generators for $\text{sdiff}M$ are much needed to investigate the representation theory of $\text{sdiff}M$, which will reveal many useful properties of $\text{SDiff}M$, as the representations of $\text{Diff}S^1$ provide important

information to study conformal field theories, but unfortunately it is a complete mystery.

In general, we are not able to express the generators of the Lie algebra $\text{sdiff } M$ in terms of a local coordinates explicitly, but the generators for the abelian subalgebra have local expressions:

$$[L_m, L_n] = 0 \quad \text{for} \quad L_n := x^n y^{-n-1} \partial_x + x^{n-1} y^{-n} \partial_y, \quad (2.22)$$

where (x, y) is a set of Riemann normal coordinates. These generators are well-defined only away from the coordinate origin.

2.6. Variational principle in the h -decomposition

For a given gravitational action \mathcal{S} the variation with respect to arbitrary infinitesimal change of metric is given by

$$\delta \mathcal{S} = \int_M \sqrt{|g|} T_{\mu\nu} \delta g^{\mu\nu}. \quad (2.23)$$

To derive equations of motion $\delta g^{\mu\nu}$ is any metric deformation in the configuration space and $\delta \mathcal{S} = 0$ is required, but to derive stress-energy tensor $\delta g^{\mu\nu}$ is only along the symmetry directions. In the latter case, $T_{\mu\nu}$ can be identified as a stress-energy tensor. Thus if \mathcal{S} is diffeomorphism invariant, equations of motion are given by $T_{\mu\nu} = 0$ and $\nabla^\mu T_{\mu\nu} = 0$.

But if a theory defined by an action of $\mathcal{S}(h_{\mu\nu}, \dots)$ is not manifestly invariant under $\text{Diff } M$ and the “metric” satisfies $h^{\mu\nu} \delta h_{\mu\nu} = 0$ always under $\text{Diff } M$ as in the h -decomposition, then for equations of motion we could require

$$T_{\mu\nu}^{(h)} = \frac{1}{2} T^{(h)} h_{\mu\nu} \quad (2.24)$$

for some function $T^{(h)}$ yet to be determined such that $\delta \mathcal{S} = \frac{1}{2} \int_M \sqrt{|h|} T^{(h)} h_{\mu\nu} \delta h^{\mu\nu} = 0$. Thus it defines a modified variational principle. If we regard the LHS of eq.(2.24) as sort of energy-momentum, it is neither traceless, nor covariantly conserved unless $T^{(h)}$ is constant such that

$$\nabla^{(h)\mu} T_{\mu\nu}^{(h)} = \frac{1}{2} \partial_\nu T^{(h)}. \quad (2.25)$$

This does not necessarily mean that the theory is ill-defined, but it simply implies that $h_{\mu\nu}$ is not a metric. We can always define a new object⁴ $T_{\mu\nu} \equiv T_{\mu\nu}^{(h)} - \frac{1}{2} T^{(h)} h_{\mu\nu}$, which is nothing but

⁴Such $T_{\mu\nu}$ also shows up in [8].

the stress-energy tensor with respect to the metric $g_{\mu\nu}$ and the equation of motion now reads $T_{\mu\nu} = 0$. Thus requiring $\text{SDiff}M$ -invariance only does not necessarily contradict to the general covariance.

3. Liouville Gravity

3.1. Generic $\text{SDiff}M$ structure

Liouville gravity is an induced gravity of a two-dimensional system. For an arbitrary metric before any gauge fixing, the action has a nonlocal form

$$\mathcal{S}_L = \frac{c}{96\pi} \iint_M R \Delta^{-1} R + \frac{c}{24\pi} \int_M \Lambda. \quad (3.1)$$

More precisely,

$$\mathcal{S}_L = \frac{c}{96\pi} \int_M d^2x_1 d^2x_2 \sqrt{|g(x_1)|} \sqrt{|g(x_2)|} R(x_2) K(x_1, x_2) R(x_1) + \frac{c}{24\pi} \int_M d^2x \sqrt{|g(x)|} \Lambda, \quad (3.2)$$

where $\sqrt{|g(x_1)|} \Delta_1 K(x_1, x_2) = \delta(x_1 - x_2)$ and scalar curvature $R = \Delta \Phi = -g^{\mu\nu} \nabla_\mu \nabla_\nu \Phi$ for some Φ which is not necessarily globally defined. Note that K , or Φ , is not uniquely defined but only up to zero modes that are nothing but harmonic functions defined on manifold M . Since there is no such a globally defined nontrivial (*i.e.* not constant) harmonic function on a compact Riemannian manifold without boundary, one can always express eq.(3.1) locally. Otherwise, the local action is not uniquely defined. Nevertheless, as we shall show below, we can interpret this freedom of non-local action as a symmetry of equations of motion. It is also worth while to mention that this action is related to the $\text{SL}(2, \mathbb{R})$ Chern-Simons theory[15] and a complete local form of this action (up to a surface term) for the Euclidean signature is given in the same paper.

If we choose the conformal gauge $g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu}$ for $\text{Diff}M$, \mathcal{S}_L reduces to the well-known Liouville action. Since ϕ transforms like a scalar, the expression $g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu}$ is covariant under conformal diffeomorphisms and invariant under isometries. As a result, the Liouville action is invariant under conformal diffeomorphisms as well as isometries. Therefore, the conformal gauge is not a true gauge fixing condition for $\text{Diff}M$, but it rather reduces $\text{Diff}M$ to the space of conformal diffeomorphisms \mathcal{W}_c and isometries. Note that the isometry group is part of $\text{SDiff}M$. Since the isometry group of a pseudo-sphere is $\text{ISO}(1,1)$ (or $\text{ISO}(2)$ for a sphere) which is isomorphic to $\text{SL}(2, \mathbb{R})$, this explains the origin of $\text{SL}(2, \mathbb{R})$ symmetry in [5].

Normally, we take it for granted that ϕ is a scalar under $\text{Diff}M$. However, if ϕ does not transform like a scalar, but satisfies $\mathcal{L}_v(\phi) = \nabla_\mu v^\mu$, we can show that the expression $g_{\mu\nu} = e^{2\phi}\eta_{\mu\nu}$ is in fact covariant under the general coordinate transformations, despite $\eta_{\mu\nu}$ being constant. Furthermore, this expression is actually $\text{SDiff}M$ -invariant because $\mathcal{L}_v(\phi) = 0$ for $\nabla_\mu v^\mu = 0$. Therefore, in this case the conformal gauge is not a true gauge fixing condition for $\text{Diff}M$ and $\eta_{\mu\nu}$ is not a constant metric tensor but constant expression for $h_{\mu\nu}$ in the h-decomposition.

This also indicates that the quantization of the Liouville gravity in the conformal gauge should be more involved and perhaps Batalin-Vilkovisky quantization may be useful due to the secondary gauge symmetry of \mathcal{W}_c or $\text{SDiff}M$ that needs to be further fixed[16]. (Further work is in progress in this direction.)

In fact we can do more generally. Using any generic decomposition of a metric $g_{\mu\nu} = e^{2\phi}\sigma_{\mu\nu}$, we can derive an equivalent Liouville action from eq.(3.2) with respect to the background “metric” $\sigma_{\mu\nu}$ as

$$\mathcal{S}_L = \mathcal{S}_{NL}(\sigma) + \mathcal{S}_{LL}(\sigma, \phi), \quad (3.3)$$

where

$$\mathcal{S}_{NL}(\sigma) = \frac{c}{96\pi} \int_M \sqrt{|\sigma|} \sqrt{|\sigma|} R(\sigma) K R(\sigma) \quad (3.4)$$

$$\mathcal{S}_{LL}(\sigma, \phi) = \frac{c}{24\pi} \int_M d^2x \sqrt{|\sigma|} \left(\sigma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \phi R(\sigma) + \Lambda e^{2\phi} \right). \quad (3.5)$$

If $\Lambda = 0$, $\mathcal{S}_{LL}(\sigma, h)$ has an additional hidden symmetry that is not manifest at the action level, but it is a symmetry of the equation of motion. One can always shift ϕ by harmonic functions. This is a reminder of the fact that the kernel K is only defined up to zero modes that are harmonic functions. If $\partial M = 0$, then this symmetry corresponds to shifting the action merely by a constant. It can be actually shown that \mathcal{S}_{LL} is a well-defined integration over M under the $\text{SDiff}M$ gauge symmetry in both h- and g-decomposition, using equations of motion.

In the h-decomposition, since $h_{\mu\nu}$ does not really transform like a metric tensor, $\mathcal{S}_{LL}(h, \phi)$ is not manifestly $\text{Diff}M$ -invariant. Using $\nabla_\mu v^\mu = 0$ and eqs.(2.15-2.16), it can be shown that it is not $\text{SDiff}M$ -invariant either in general, unless $v^\alpha \partial_\alpha \phi$ is harmonic but nonvanishing. At this moment we do not know the effect of $\nabla_\mu \nabla^\mu (v^\alpha \partial_\alpha \phi) = 0$ on $\text{SDiff}M$ precisely yet, so we will just assume that there exist such cases. (For more discussion, see the last section.) Thus with this constraint $\mathcal{S}_{LL}(h, \phi)$ is $\text{SDiff}M$ -invariant. As a result, \mathcal{S}_{NL} is also $\text{SDiff}M$ -invariant but not $\text{Diff}M$ -invariant. Note that $\mathcal{S}_{LL}(h, \phi)$ is invariant under Weyl rescaling up to equations of

motion.

If we want a metric in which the Liouville field transforms like a scalar under $\text{SDiff}M$, then we can take the g-decomposition. Now the action becomes $\mathcal{S}_{LL}(\hat{g}, \hat{\phi})$ and it is manifestly $\text{SDiff}M$ -invariant without imposing any extra condition. However, in this case we cannot set $\hat{g}_{\mu\nu}$ to $\eta_{\mu\nu}$ and there is no conformal symmetry.

3.2. Remarks on $\gamma^{\mu\nu} \equiv \sqrt{|g|}g^{\mu\nu}$ case

In [7, 8] it is argued that there exists an action of the form eq.(3.1) that depends only on the Weyl invariant combination $\gamma^{\mu\nu} \equiv \sqrt{|g|}g^{\mu\nu}$ rather than each separately but differs by local terms. Here we shall adopt the same idea but we impose the covariant condition so that the follows are not necessarily the same. Note that $\gamma^{\mu\nu}$ is a tensor density so that it is not a metric tensor. And it does satisfy the criterion for the h-decomposition because $\gamma^{\mu\nu}\delta_f\gamma_{\mu\nu} = 0$ for any vector field in $\text{diff}M$ and $\delta_f\sqrt{|g|} = \sqrt{|g|}\nabla_\mu v^\mu$. Thus it is a case of the h-decomposition.

In 2-d we have an identity

$$\sqrt{|g|}R = R(\gamma) + \Delta^{(\gamma)} \ln \sqrt{|g|}, \quad (3.6)$$

where $R(\gamma)$ is the “scalar curvature” computed in terms $\gamma_{\mu\nu}$ as if $\gamma_{\mu\nu}$ were a metric tensor and $\Delta^{(\gamma)} = -\partial_\mu(\gamma^{\mu\nu}\partial_\nu) = \sqrt{|g|}\Delta$. This also shows that in general $\sqrt{|g|}R$ is not Weyl-invariant in 2-d because the second term in the RHS of eq.(3.6) is not Weyl invariant unless the change $\ln \sqrt{|g|}$ is harmonic. Locally under coordinate transformations, $R(\gamma)$ is not invariant but transforms like

$$\begin{aligned} \sqrt{|\gamma|}R(\gamma) &= R(\gamma) \rightarrow J^{-1} \left(R(\gamma) - \Delta^{(\gamma)} \ln J^{-1} \right), \\ \sqrt{|\gamma|}\Delta^{(\gamma)} \ln \sqrt{|g|} &\rightarrow J^{-1}\Delta^{(\gamma)} \ln \left(J^{-1}\sqrt{|g|} \right), \end{aligned} \quad (3.7)$$

where J is the jacobian of the coordinate transformation. Note that $R(\gamma)$ does not transform like a scalar, unless J is a harmonic function.

Anyhow, eq.(3.3) can be decomposed as

$$\mathcal{S}_L = \mathcal{S}_W(\gamma) + \mathcal{S}_{NW}(\gamma, \ln \sqrt{|g|}), \quad (3.8)$$

$$\mathcal{S}_W = \frac{c}{96\pi} \int_M d^2x_1 d^2x_2 R(\gamma(x_2)) K(x_1, x_2) R(\gamma(x_2)), \quad (3.9)$$

$$\mathcal{S}_{NW}(\gamma, \ln \sqrt{|g|}) = \frac{c}{96\pi} \int_M d^2x \left(\gamma^{\mu\nu} \partial_\mu \ln \sqrt{|g|} \partial_\nu \ln \sqrt{|g|} + 2 \ln \sqrt{|g|} R(\gamma) + 4\Lambda \sqrt{|g|} \right), \quad (3.10)$$

where \mathcal{S}_W is Weyl invariant, but the integrand is not a scalar density.

Nevertheless, we can check how \mathcal{S}_W behaves under $\text{SDiff}M$. In fact, we can take a short cut, using the generic property of h-decomposed metric instead of directly computing the Lie derivative of \mathcal{S}_W . One can easily show that \mathcal{S}_L and \mathcal{S}_{NW} are $\text{SDiff}M$ -invariant if $v^\alpha \partial_\alpha \ln \sqrt{|g|}$ is harmonic so that \mathcal{S}_W has to be $\text{SDiff}M$ -invariant. If $\Lambda = 0$, \mathcal{S}_{NW} is also Weyl invariant up to equations of motion.

The rationale behind this approach is that the combined symmetry of area preserving diffeomorphisms and Weyl rescaling provides the same amount of gauge degrees of freedom as diffeomorphism invariance. This is because we can trade off $\mathcal{W}_c \in \text{Diff}M$ with Weyl rescalings. To realize this idea one need to show that any change of coordinate basis can be achieved by combined coordinate transformation by $\text{SDiff}M$ and Weyl rescaling so that the integration of \mathcal{S}_W can be consistently defined over M . If we impose the covariant condition $\nabla_\mu v^\mu = 0$, this can be accomplished with the help of equations of motion.

4. Geometric Liouville Gravity

4.1. Lagrangian and Duality

For argument's sake we shall start from the dual form of Liouville gravity action, then later decompose into the geometric Liouville action written in terms of $\text{SDiff}M$ variables in the g-decomposition to recover the results in [3]. To obtain the result in the h-decomposition we simply redefine the variables. However, equations of motion will have a different form due to the different variational principle.

For this purpose, we enlarge the configuration space to the space of all metrics for arbitrary genus surfaces and take $\text{Diff}M$ as gauge symmetry. Following [3], we relate the scalar field Φ such that $\Delta\Phi = R$ to the affine spin connection ω by

$$\omega = - * d\Phi \quad \text{or} \quad \omega_\mu = \epsilon_{\mu\nu} \partial^\nu \Phi. \quad (4.1)$$

In this sense, Φ and ω are dual to each other.

Now $\delta\omega = 0$ is satisfied so that the curvature two-form can be written as

$$\hat{R} = (d + \delta)\omega = \Delta * \Phi, \quad (4.2)$$

where Δ is the Laplace-Beltrami operator. Then the first term in eq.(3.1) can be rewritten as

$$\int_M \Delta^{-1} \hat{R} \wedge * \hat{R} = \int_M (d + \delta)^{-1} \omega \wedge *(d + \delta) \omega. \quad (4.3)$$

Using the Hodge dual property of scalar product on compact oriented manifold, now the theory of action eq.(3.1) is locally equivalent to that of an action which can be written as a local form

$$\mathcal{S}_A = \frac{c}{96\pi} \int_M \omega \wedge * \omega + \frac{c}{24\pi} \int_M * \Lambda = \frac{c}{96\pi} \int_M d^2 x \mathbf{e} (\omega_\mu \omega^\mu + 4\Lambda) \quad (4.4)$$

with gauge fixing conditions

$$\delta \omega = -\nabla_\mu \omega^\mu = 0, \quad (4.5)$$

$$n^\mu w_\mu = 0 \quad \text{on } \partial M \text{ for } n^\mu \perp \partial M, \quad (4.6)$$

where $\mathbf{e} = \sqrt{|g|}$ is the zweibein volume element. This gauge fixing condition, which is motivated by the area preserving condition of $\text{sdiff} M$, fixes the local Lorentz symmetry in the frame space, that is, $\text{SO}(1,1)$ (or $\text{SO}(2)$ depending on the signature) in this case. Due to this gauge fixing, the ambiguity related to the zero modes of Δ in the original Liouville action \mathcal{S}_L is no longer present because \mathcal{S}_A is now invariant under $\omega \rightarrow \omega + \lambda_H$, where one-form λ_H corresponds to the zero mode shift of Φ . Thus, \mathcal{S}_A is a uniquely defined local action corresponding to the nonlocal action \mathcal{S}_L and the zero mode ambiguity is now identified as a symmetry.

Notice that this gauge fixing condition is invariant under $\text{Diff} M$ and can be imposed globally on M , although ω itself is not globally defined on M . Thus the metric is still not constrained. Metric gauge fixing should be imposed independently. In the frame space, although $\nabla^\mu \omega_\mu$ is invariant under only global $\text{SO}(1,1)$ (or $\text{SO}(2)$), but in fact $\nabla^\mu \omega_\mu = 0$ is preserved under global $\text{GL}(2, \mathbb{R})$. Under local $\text{SO}(1,1)$, the gauge fixing condition is preserved only up to a harmonic function. Note that local Lorentz transformations and $\text{Diff} M$ do not commute. In general, \mathcal{S}_A in eq.(4.4) is more general than the original Liouville gravity because the equivalence to the Liouville gravity action is true only if $\delta \omega = 0$. In some sense this is due to the nonlocality of the original action. Later we shall abandon this gauge fixing condition and investigate \mathcal{S}_A itself.

For the time being, zweibeins e_μ^a will be considered to be the only fundamental variables, rather than treating (e, ω) independently as is often done in a gauge theory formulation of gravity. Thus the affine spin connection is always computed in terms of zweibeins, so we do not need to impose the torsion-free condition as a constraint. In this sense, the action takes a form analogous to gauge theory action $\int F^2$.

Under $\text{Diff}M$ the Lagrangian in eq.(4.4) changes according to the Lie derivative as

$$\mathcal{L}_v(\omega \wedge * \omega) = d(i_v \omega \wedge * \omega - \omega \wedge i_v * \omega). \quad (4.7)$$

Although it is a total derivative, if $\partial M \neq 0$, this does not necessarily vanish when integrated over M . But it vanishes if $v^\alpha \in \text{sdiff}M$.

To compare to the Liouville case, we rewrite in terms of Φ in eq.(4.1), then the action eq.(4.4) reads

$$\mathcal{S}_A = \frac{c}{96\pi} \int_M d^2x \sqrt{|g|} [g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \lambda (R - \Delta \Phi) + 4\Lambda], \quad (4.8)$$

where λ is the Lagrange multiplier for the constraint $R = \Delta \Phi$ and Φ is defined only up to a harmonic function. Of course, this action can also be derived directly from eq.(3.1) but then the zero mode ambiguity may not be clearly resolved.

Variation with respect to Φ leads to

$$\lambda = 2\Phi$$

so that we can rewrite the action for this value as

$$\mathcal{S}_A = \frac{c}{48\pi} \int_M d^2x \sqrt{|g|} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \Phi R + 2\Lambda \right), \quad (4.9)$$

which leads to equivalent equations of motion. Here, the Liouville field behaves like a real scalar field except that the kinetic energy has an opposite sign compared to eq.(4.8).

Note that there is no exponential potential term compared to the usual Liouville action so that Φ is not the usual Liouville field. To identify the usual Liouville field the metric needs to be decomposed. To reproduce the result in [3] we take the g-decomposition, which produces manifestly $\text{SDiff}M$ covariant objects. Then \mathcal{S}_A decomposes as

$$\mathcal{S}_A = \mathcal{S}_G(\hat{g}) + \mathcal{S}_{AL}(\hat{g}, \hat{\phi}), \quad (4.10)$$

$$\mathcal{S}_G(\hat{g}) = \frac{c}{48\pi} \int_M d^2x \sqrt{|\hat{g}|} \left(-\frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \Phi(\hat{g}) \partial_\nu \Phi(\hat{g}) + \Phi(\hat{g}) R(\hat{g}) \right), \quad (4.11)$$

$$\mathcal{S}_{AL}(\hat{g}, \hat{\phi}) = \frac{c}{24\pi} \int_M d^2x \sqrt{|\hat{g}|} \left(\hat{g}^{\mu\nu} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} + \hat{\phi} R(\hat{g}) + \Lambda e^{2\hat{\phi}} \right). \quad (4.12)$$

$\mathcal{S}_G(\hat{g})$ is equivalent to the geometric (Liouville) action in [3], whilst $\mathcal{S}_{AL}(\hat{g}, \hat{\phi})$ is the usual Liouville action in this metric decomposition. Note that \mathcal{S}_G is manifestly $\text{SDiff}M$ -invariant, but

not $\text{Diff}M$ -invariant because $\hat{\phi}$ is absent. As a result, \mathcal{S}_{AL} is not $\text{Diff}M$ -invariant but $\text{SDiff}M$ -invariant. In the h-decomposition, $\mathcal{S}_G(h)$ is Weyl invariant under any Weyl transformation of the metric $g_{\mu\nu}$ because $h_{\mu\nu}$ does not change but only ϕ changes. But in the g-decomposition $\mathcal{S}_G(\hat{g})$ is not Weyl invariant in the usual sense. However, there is an analogous symmetry for $\mathcal{S}_G(\hat{g})$ under $\Phi(\hat{g}) \rightarrow \Phi(\hat{g}) + \delta\Phi$, which leaves $\mathcal{S}_G(\hat{g})$ invariant up to equations of motion. This is due to the zero mode ambiguity of the original action.

One can easily observe that in fact \mathcal{S}_G contains all the classical information about \mathcal{S}_A for $\Lambda = 0$. Since the tree level cosmological constant is not really important toward quantum theory in this context, it is good enough to use \mathcal{S}_G as a whole classical action. \mathcal{S}_{AL} simply generalizes to include $\Lambda \neq 0$. This is why \mathcal{S}_G is often just ignored by imposing a constraint on $\hat{\Phi}$ and \mathcal{S}_{AL} is used to describe the Liouville theory in the third kind of decomposition $g_{\mu\nu} = e^{2\varphi}\tilde{g}_{\mu\nu}$, in which $\tilde{g}_{\mu\nu}$ transforms like a metric tensor. Here, we now realize that this is just one special case and we can fix $\text{Diff}M$ in many different ways. We can select either \mathcal{S}_G or \mathcal{S}_{AL} to describe the classical theory in any metric decomposition.

In the g-decomposition, \mathcal{S}_G is equivalent to the geometric action constructed in [3]. If we represent the action in terms of ω , then the action is an analog of the vorticity hamiltonian in fluid dynamics, where the coupling constant c takes the role of the density and $\omega^\mu\partial_\mu \in \text{sdiff}M$.

In the g-decomposition, the stress-energy tensor works the same way as in $\text{Diff}M$ case. But, to derive a conserved quantity in the h-decomposition, now we should use the modified variational principle with respect to $\text{Diff}M$. Then from eq.(4.11) we obtain

$$T_{\mu\nu}^{(h)} = \partial_\mu\Phi(h)\partial_\nu\Phi(h) - \frac{1}{2}h_{\mu\nu}h^{\alpha\beta}\partial_\alpha\Phi(h)\partial_\beta\Phi(h) + \frac{1}{2}h_{\mu\nu}\Delta^{(h)}\Phi(h) = \frac{1}{2}T^{(h)}h_{\mu\nu}, \quad (4.13)$$

where $h^{\mu\nu}T_{\mu\nu}^{(h)} = T^{(h)}$ determines $T^{(h)} = \Delta^{(h)}\Phi(h) = R(h)$. Thus $T_{\mu\nu}^{(h)}$ is neither traceless nor covariantly conserved. As we alluded before, we can always define $T_{\mu\nu}^{\text{phys}} \equiv T_{\mu\nu}^{(h)} - \frac{1}{2}R(h)h_{\mu\nu} = 0$, which implies an equivalent result in the $\text{Diff}M$ case by a simple Weyl transformation and vice versa. Therefore, classically it does not contradict to the general covariance.

On the other hand, we can also regard T in the above as a gauge parameter such that $T = R(h)$ is not an identity but an equation of motion. Fixing T corresponds to fixing the gauge degrees of freedom of $\text{Diff}M$. For example, $T = 0 = R(h)$ leads to the $\text{SDiff}M$ invariance of $\mathcal{S}_{AL}(h, \phi)$.

In the g-decomposition, T must vanish, leaving $R(\hat{g}) = 0$. Then $R(g) = -2\Lambda$. Since $\hat{g}_{\mu\nu}$ behaves like a metric tensor under $\text{SDiff}M$, we can imitate the $\text{Diff}M$ case to solve this equation.

Nonetheless, it still leaves two components of $g_{\mu\nu}$ undetermined, namely, $\hat{\phi}$ and one of $\hat{g}_{\mu\nu}$. In some cases, $\hat{\phi}$ and the remaining component of $\hat{g}_{\mu\nu}$ may not be separable in $g_{\mu\nu}$ in practice. One good example is $\hat{g}_{\mu\nu} = e^{2\hat{\rho}}\eta_{\mu\nu}$ gauge fixing. This does not necessarily indicate quantization in the g-decomposition will be the same as the DiffM case because the transformation law for $\hat{g}_{\mu\nu}$ is not really independent from $\hat{\phi}$ due to the area preserving condition.

4.2. More about $\delta\omega = 0$

Since this gauge fixing of local Lorentz invariance is the key to relate \mathcal{S}_A to \mathcal{S}_L , it deserves some more attention. We already pointed out that this gauge fixing is necessary due to the nonlocality of \mathcal{S}_L .

First, we may attempt to find if there is any equivalent gauge fixing condition acting directly on zweibein themselves. Using

$$[\nabla_\alpha, \nabla_\nu] e_\mu^a = -e_\lambda^a R_{\mu\alpha\nu}^\lambda,$$

we obtain

$$\begin{aligned} \nabla_\mu \omega^{\mu a}_b &= -E_b^\nu \nabla^\mu \nabla_\mu e_\nu^a \\ &= -E_b^\nu \nabla_\nu \nabla^\mu e_\mu^a - \frac{1}{2} \delta_b^a R - E_b^\nu \nabla^\mu (\nabla_\mu e_\nu^a - \nabla_\nu e_\mu^a). \end{aligned} \quad (4.14)$$

Since $a \neq b$, R -term vanishes. Now one may be tempted to conclude that $\nabla^\mu e_\mu^a = 0$ and $\nabla_\mu e_\nu^a - \nabla_\nu e_\mu^a = 0$ seem to be sufficient conditions to satisfy $\nabla_\mu \omega^\mu = 0$, but this is more than what we need for gauge fixing. In fact the above are nothing but $\delta e^a = 0 = de^a$ so that e^a becomes a harmonic one-form. Using the torsion constraint, we obtain $\omega_b^a \wedge e^b = 0$. In this case one can easily see that $de^a = 0$ implies $\omega_\mu = 0$ and that $g^{\mu\nu} \omega_\mu \omega_\nu = 0$, *i.e.* the action itself vanishes. Thus it does not seem to be possible to obtain any simple condition directly on zweibeins.

There is another way to check this gauge fixing condition. A zweibein basis of a frame changes under DiffM, so we need to understand the global property of such a gauge fixing more carefully. Since $\delta\omega = 0$, locally we can obtain $\omega = \delta W$ for some two-form $W = \frac{1}{2} W_{\mu\nu} dx^\mu dx^\nu$. Let $W_{\mu\nu} = \epsilon_{\mu\nu} \Phi$, then

$$\omega = \delta \hat{\Phi}, \quad (4.15)$$

where $\hat{\Phi} \equiv * \Phi$. Comparing to the curvature two-form $\hat{R} = d\omega$, we obtain

$$\hat{R} = d\delta \hat{\Phi}. \quad (4.16)$$

Globally ω is determined up to co-closed one-form λ such that $\delta\lambda = 0$, hence $\omega = \delta\hat{\Phi} + \lambda$. Eq.(4.16) in turn implies λ is also a closed one-form. so that λ is in fact a harmonic one-form. Therefore, $\delta\omega = 0$ gauge is equivalent to choosing λ to be a harmonic one-form.

Now let us check if $\delta\omega = 0$ uniquely fixes all the gauge degrees of freedom or there are any secondary gauge degrees of freedom which leave λ invariant. From eq.(4.16) we can always add an exact one-form dF , which makes the complete decomposition of $\omega = \delta\hat{\Phi} + \lambda_H + dF$, where λ_H denotes a harmonic one-form. The gauge fixing condition implies $\Delta F = 0$ so that F must be a harmonic function and dF is a harmonic one-form. Thus in general $\delta\omega = 0$ does not fix λ uniquely. In other words, the gauge fixing condition is preserved by the change of Laplacian of a harmonic function⁵. In the case of $g_{\mu\nu} = e^{2\rho}\eta_{\mu\nu}$, one can easily show that this actually corresponds to local $SO(1,1)$ (or $SO(2)$) symmetry.

On a compact Riemannian manifold without boundary $\delta\omega = 0$ does fix the gauge completely (up to a discrete set of harmonic one-forms depending on the topology) because the above decomposition is unique according to the Hodge decomposition theorem[17], and that a harmonic function is necessarily a constant so that $dF = 0$. Thus $\omega = \delta\hat{\Phi} + \lambda_H$ is the unique decomposition on a compact manifold without boundary. The symmetry corresponding to F is no longer local, but global $SO(1,1)$ (or $SO(2)$) symmetry.

4.3. (e, ω) independently

One can also think about

$$\mathcal{S}_A(e, \omega) = c \int_M \omega \wedge * \omega + c\Lambda \int_M e^1 \wedge e^2 \quad (4.17)$$

as a defining action for a gravitational theory. Compared to the first-order formalism, there is no explicit derivative terms in this action because ω is no longer directly related to zweibeins. Thus in particular there is no explicit kinetic energy term. We only have “mass” terms, if we regard (e, ω) as gauge fields of $ISO(1,1)$ (or $SL(2, \mathbb{R})$). In particular, classical equations of motion are just $\omega = 0$ and $\Lambda = 0$ which can be regarded as unbroken phase of a gravitational theory *à la* Witten[18] because the classical action vanishes for these values and there are no dynamical degrees of freedom. This indicates that perhaps $\mathcal{S}_A(e, \omega)$ may define an integrable theory, if not topological. The integrability of this theory may not be surprising at the end

⁵Such a situation happens in QED too. The Lorentz gauge condition is invariant if the gauge parameter is a harmonic function.

because after all it is related to the Liouville theory which is integrable.

Here, in principle ω^μ can be any vector on M . The reason why $\text{SDiff}M$ becomes relevant in [3] is because $\delta\omega = 0$ is imposed as a constraint so that $\omega^\mu \in \text{sdiff}M$ as a vector field on M . Then \mathcal{S}_A becomes a hamiltonian which describes how M deforms keeping the area of M fixed. Once $\delta\omega = 0$ is imposed, this is dual to the usual Liouville action as we constructed and there is an additional ‘‘Weyl’’ symmetry: $\omega \rightarrow \omega + *d\rho$ for some function ρ . But this constraint is not essential in general for \mathcal{S}_A . However, to define a reasonable gravitational theory on a Riemannian manifold M , there is one constraint we must impose. This is the torsion-free constraint $de + \omega e = 0$ and it also identifies ω as the affine spin connection.

Thus imposing the following constraint, we recover the dynamics:

$$\mathcal{D}_\mu e_\nu^a := \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\lambda e_\lambda^a + \omega_\mu^a{}_b e_\nu^b = 0, \quad (4.18)$$

where \mathcal{D}_μ is the covariant derivative acting both on M and on the frame space. This constraint recovers the previous relation between spin connection and zweibeins, eq.(2.10).

If we introduce only the torsion-free constraint into the action, we obtain

$$\mathcal{S}_A(e, \omega) = c \int_M (\omega \wedge *\omega + \lambda(de + \omega \wedge e)) + c\Lambda \int_M e^1 \wedge e^2. \quad (4.19)$$

Variation with respect to ω derives $\lambda e = -*\omega$ and the rest of equations of motion are simply

$$\omega = 0, \quad de = 0, \quad \Lambda = 0.$$

Compared to the Liouville case in which $R = -2\Lambda$, we obtain a different result, unless $\Lambda = 0$. This is why we need $\delta\omega = 0$ for \mathcal{S}_A to be related to the Liouville gravity.

5. Conclusion and Discussions

We have provided a general framework to construct $\text{SDiff}M$ -invariant gravitational theories in two-dimensions, which are not necessarily manifestly $\text{Diff}M$ -invariant. From $\text{Diff}M$'s point of view, fundamental field variables are no longer globally defined, which is not unusual in gauge theories. Two different ways of defining such field variables are introduced: h- and g-decomposition. In the h-decomposition, it is necessary to impose equations of motion to define a consistent integration on M , whilst the integration in the g-decomposition is well-defined

without imposing any conditions as in $\text{Diff}M$ cases if an integrand is a scalar density with respect to $\text{SDiff}M$.

So far as Liouville gravity is concerned, we have shown that there is $\text{SDiff}M$ invariant subsystem which contains sufficient information about the original system. In this sense it does not violate the general covariance at the level of classical equations of motion. Now, one may ask if there are any merits to use $\text{SDiff}M$ invariant system rather than $\text{Diff}M$ invariant system, since they come out to be equivalent classically. Nevertheless, we expect a real difference may show up in quantum theories, particularly in which any dilaton degrees of freedom are completely frozen. To describe a physical system we are required to fix all the gauge degrees of freedom so that in principle we can allow a physical gravitational system in which all conformal degrees of freedom in $\text{Diff}M$ are spontaneously broken as well as Weyl symmetry. And the “physical” dilaton may incorporate not only Goldstone modes of Weyl symmetry but also those of conformal symmetry in $\text{Diff}M$. In other words, we can define a (massive) dilaton as a (pseudo-)Goldstone boson of $\text{Weyl} \otimes \text{Diff}M$ to $\text{SDiff}M$ symmetry breaking. To describe quantum physics of such a dilaton, the formalism we described in this paper should be useful. So we expect that the key to resolve the mystery of the massive dilaton may reside in this framework.

Also $\text{SDiff}M$ invariance provides a framework to describe intrinsically a theory defined on a manifold with boundary without introducing a boundary term. This inevitably addresses an issue of the energy-momentum conservation at the boundary, but as we pointed out the physical energy-momentum can always be defined to be conserved.

Many questions remain to be answered. For example, in the g-decomposition $\text{SDiff}M$ invariance is manifest by construction, whilst in the h-decomposition we need an extra constraint to show $\text{SDiff}M$ invariance in the Liouville case. It is not clear how this extra constraint $\nabla^\mu \nabla_\mu v^\alpha \partial_\alpha \phi = 0$ restricts $\text{SDiff}M$. In the simpler case of isometry group, this condition requires $R(h) \propto e^{2\phi}$ so that we can anticipate that it may actually restrict the form of $h_{\mu\nu}$. In other words, the Liouville action is not a good candidate to be $\text{SDiff}M$ invariant in the h-decomposition. It would be interesting to know if there is a modified action that is invariant under $\text{SDiff}M$ without any further constraint.

It is also necessary to know how to quantize such a $\text{SDiff}M$ invariant system consistently. Our hope is that there may be a generation of dilaton potential in this approach because there is no symmetry which prohibits this from happening. From the conventional point of view, we can speculate that the trace of the graviton may be absorbed into the dilaton to provide

dilaton mass. We hope further investigation in this direction reveals more physical roles of the area preserving diffeomorphism in gravity in general.

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